

# Dynamics of $SU(N)$ Supersymmetric Gauge Theory

Michael R. Douglas

*and*

Stephen H. Shenker

Dept. of Physics and Astronomy

Rutgers University

Piscataway, NJ 08855-0849

`mrd@physics.rutgers.edu`

`shenker@physics.rutgers.edu`

We study the physics of the Seiberg-Witten and Argyres-Faraggi-Klemm-Lerche-Theisen-Yankielowicz solutions of  $D = 4$ ,  $\mathcal{N} = 2$  and  $\mathcal{N} = 1$   $SU(N)$  supersymmetric gauge theory. The  $\mathcal{N} = 1$  theory is confining and its effective Lagrangian is a spontaneously broken  $U(1)^{N-1}$  abelian gauge theory. We identify some features of its physics which see this internal structure, including a spectrum of different string tensions. We discuss the limit  $N \rightarrow \infty$ , identify a scaling regime in which instanton and monopole effects survive, and give exact results for the crossover from weak to strong coupling along a scaling trajectory. We find a large hierarchy of mass scales in the scaling regime, including very light  $W$  bosons, and the absence of weak coupling. The light  $W$ 's leave a novel imprint on the effective dual magnetic theory. The effective Lagrangian appears to be inadequate to understand the conventional large  $N$  limit of the confining  $\mathcal{N} = 1$  theory.

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## 1. Introduction

Over the last year and a half, revolutionary progress in understanding the dynamics of four-dimensional supersymmetric gauge theories has been made by Seiberg and collaborators [1]. One spectacular result is the exact low-energy effective Lagrangian for  $\mathcal{N} = 2$  supersymmetric  $SU(2)$  gauge theory obtained by Seiberg and Witten [2]. The  $\mathcal{N} = 2$  vector multiplet contains an adjoint scalar whose non-zero vacuum expectation value (vev) breaks  $SU(2)$  to  $U(1)$ . For large vev compared to the scale  $\Lambda$  set by the gauge coupling, one can write an effective Lagrangian in terms of a  $U(1)$  gauge multiplet. For small vev, in the past we would have expected  $SU(2)$  color confinement and a very different (and inaccessible) description.

Surprisingly, it turned out that the effective theory is a  $U(1)$  gauge theory for any expectation value, and the naive unbroken  $SU(2)$  regime is not present at all. Instead, two singularities of the effective action appear in the strong coupling regime. At one of these, the monopole visible in the semiclassical treatment becomes arbitrarily light. The effective Lagrangian is again a  $U(1)$  gauge theory but now written in terms of a vector multiplet containing the dual (magnetic) gauge field with the standard local coupling to the monopole, as well as a hypermultiplet describing the monopoles. The other singularity is isomorphic to this, with the role of the monopole taken by a charge  $(1, 1)$  dyon.

Thus the  $\mathcal{N} = 2$  theory does not confine. Surprising as this result may be, it does not drastically contradict previous expectations, mostly because of the presence of the massless scalar. Theories which do not have such a scalar and are widely believed to be similar to pure (bosonic) gauge theory are  $\mathcal{N} = 1$  SYM gauge theory as well as supersymmetric QCD. These theories should confine, and Seiberg and Witten showed that this could be explained in  $\mathcal{N} = 1$  theories obtained by adding an  $\mathcal{N} = 2$  breaking mass term: it is the result of monopole condensation.

These results have been extended by Argyres and Faraggi [3] and by Klemm, Lerche, Theisen and Yankielowicz [4] to pure  $SU(N)$  gauge theory.\* The  $\mathcal{N} = 2$  theory now has  $N - 1$  moduli (say, the eigenvalues of the adjoint scalar vev  $\langle\phi\rangle$ ) and in the semiclassical regime the gauge symmetry is broken to  $U(1)^{N-1}$ . Each factor contains a monopole solution and by varying the moduli one can drive any of them massless. There are  $N$  points in moduli space at which  $N - 1$  monopoles becomes massless, and these points become the ground states of the associated  $\mathcal{N} = 1$  theory.

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\* It should be noted that the precise dependence of the effective action on the moduli proposed in equations (4) and (14) of the second paper in [4] is ambiguous for general  $N$ . We follow the unambiguous results in equations (6) and (9) of [3].

In this paper we continue the discussion of the physics of these theories begun in [2,3,4]. Perhaps the most striking result is the following. In the weakly coupled regime, the theory is a  $U(1)^{N-1}$  gauge theory with a discrete gauge symmetry  $S_N$  permuting the  $U(1)$  factors. This discrete gauge symmetry is spontaneously broken by the Higgs vev – for example, at a generic point in moduli space, every charged multiplet has a distinct mass. It turns out that this is true everywhere in moduli space, even at the vacua of the  $\mathcal{N} = 1$  theory. This leads to a non-trivial spectrum of light massive particles in the  $\mathcal{N} = 1$  theory and rather surprisingly, a spectrum of distinct string tensions in the different  $U(1)$  factors. Since the theory contains particles with charges in any pair of  $U(1)$  factors, the limiting (infinite distance) string tension is the lowest of these, but mesons and baryons bound with the higher string tensions will exist as sharp resonances for sufficiently small  $\mathcal{N} = 2$  breaking.

We also consider the large  $N$  limit of the theory and compare with expectations from previous work. Let us first say a few words about possible large  $N$  limits. The bare coupling constant is always taken to zero as  $g_0^2 \sim 1/N$  to get a theory with a planar diagram expansion. Now we will discuss only the leading (two derivative) effective Lagrangian, and in  $\mathcal{N} = 2$  SYM this receives no perturbative contributions beyond one loop. Since instantons are suppressed as  $e^{-N/g_0^2}$ , it may at first sound like there is little to do. However, this statement turns out to be naive.

In the  $\mathcal{N} = 2$  theory, there are two natural regimes to consider. To give all charged bosons masses of at least  $O(N^0)$ , the difference between any pair of eigenvalues of  $\phi$  must be at least  $O(N^0)$ , and the typical difference will be  $O(N)$ .<sup>\*</sup> The density of states will be  $O(N)$ . We refer to this as the *naive semiclassical regime* as monopole masses are  $O(N)$ , and instanton corrections do not survive in this limit. The cause of this is *not* only the  $e^{-N/g^2}$  suppression. Rather it is a non-trivial consequence of the need to introduce  $N$  distinct mass scales.

To make instanton corrections survive, one must take the spacing between eigenvalues to be  $O(1/N)$ . In this *scaling regime*, monopoles are light. The vacua of the  $\mathcal{N} = 1$  theory are of this form, and one can smoothly interpolate between them and a limit in which the spacings are  $\lambda/N$  with  $\lambda$  becoming large, in which the semiclassical treatment is valid. In this sense, there is no large  $N$  transition in the theory.

Since the effective theory is a spontaneously broken abelian gauge theory, the standard expectations of the large  $N$  limit – a finite mass gap,  $O(1)$  degeneracies in the particle

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<sup>\*</sup> We assume for simplicity that all eigenvalues are real. Otherwise there are configurations where the typical difference will be  $O(\sqrt{N})$ .

spectrum, and an effective  $\phi^3$  coupling of order  $1/N$  – are not at all obvious. We find that at least the last of these is violated near the massless monopole point.

The traditional definition of the large  $N$  limit takes  $N \rightarrow \infty$  before any other limits and in particular before the infinite volume limit. Our low energy effective Lagrangian should be accurate for processes involving momenta  $|p| \ll \Lambda_{eff}$ , where the scale is set both by the masses of particles we have integrated out and by the couplings of higher derivative terms we have dropped. Electrically charged particles must be integrated out, and it will turn out that the lightest of these has mass  $\sim \Lambda/N^2$ , which will make the interpretation of the limit subtle.

## 2. $SU(N)$ supersymmetric gauge theory

The  $\mathcal{N} = 2$ -supersymmetric bare Lagrangian is (in  $\mathcal{N} = 1$  superfield notation)

$$\mathcal{L} = \text{Im} \frac{N\tau_0}{4\pi} \left[ \int d^4\theta A^a \bar{A}^a + \int d^2\theta W_\alpha^a W_\alpha^a \right]. \quad (2.1)$$

$\tau_0 = 4\pi i/g_0^2 + \theta/2\pi$  is the bare gauge coupling. There are two reasons for the explicit  $N$  dependence. Of course, it is the appropriate one to weigh Feynman diagrams of Euler character  $\chi$  as  $N^\chi$ , and thus we will take the large  $N$  limit with  $\tau_0$  fixed. It is convenient even at finite  $N$ , since it cancels the explicit  $N$  in the one-loop beta function, and thus the dynamical scale  $\Lambda \sim \exp -\text{Im} \tau_0$  (at which the running coupling attains a prescribed  $O(1)$  value) will have no  $N$  dependence.

The  $\mathcal{N} = 2$  theory has an  $N - 1$  complex dimensional moduli space  $\mathcal{M}$  of vacua. In the classical theory these are parameterized by the invariant expectation values  $\text{Tr} \phi^n$  constructed from the scalar component of  $A$ .  $\phi$  must satisfy the  $D$ -flatness condition  $[\phi, \phi^\dagger] = 0$  and thus we can diagonalize it, and use as coordinates for  $\mathcal{M}$  the eigenvalues  $\phi_m$  with permutations identified. At a generic point  $\phi$  breaks the gauge symmetry to  $U(1)^{N-1}$  and a low-energy effective Lagrangian can be written in terms of multiplets  $(A_i, W_i)$ . We will use a ‘ $U(1)^N$ ’ notation in which  $1 \leq i \leq N$  and  $\sum_i A_i = 0$ . We denote the scalar component of  $A_i$  by  $a_i$ .

The  $\mathcal{N} = 2$  effective Lagrangian is determined by an analytic prepotential  $\mathcal{F}$  and takes the form

$$\mathcal{L}_{eff} = \text{Im} \frac{1}{4\pi} \left[ \int d^4\theta \partial_i \mathcal{F}(A) \bar{A}^i + \frac{1}{2} \int d^2\theta \partial_i \partial_j \mathcal{F}(A) W^i W^j \right]. \quad (2.2)$$

In the classical theory,  $\mathcal{F}_{cl}(A) = \frac{N\tau_0}{2} \sum_i (A_i - \sum A_j/N)^2$ . For large  $\phi_m$  the gauge coupling is weak at the scale of symmetry breaking and a good approximation to  $\mathcal{F}$  would be obtained by adding the one-loop quantum correction:

$$\mathcal{F}_1 = \frac{i}{4\pi} \sum_{i < j} (A_i - A_j)^2 \log \frac{(A_i - A_j)^2}{e^3 \Lambda^2}. \quad (2.3)$$

One can renormalize and define  $\Lambda$  to absorb  $\mathcal{F}_{cl}$  into this expression.\* In fact,  $\mathcal{N} = 2$  supersymmetry forbids further perturbative corrections [5]. This is not to say that perturbation theory is trivial but that it will only produce higher derivative terms in the effective Lagrangian.

The reduced (or BPS saturated) multiplets and their mass formula play a central role in the story. The  $U(1)^{N-1}$  theory has a lattice of allowed electric and magnetic charges, which we write  $q_i$  and  $h_i$ , again with  $\sum_i q_i = \sum_i h_i = 0$ . The charges of vector bosons are the vectors  $q_v = (0, \dots, 0, +1, 0, \dots, 0, -1, 0, \dots)$ . The fundamental representation (we use the convenient name ‘quark’) are  $q_q = (0, \dots, 0, 1, 0, \dots, 0) - (1/N, \dots, 1/N)$ . The theory contains ’t Hooft-Polyakov monopoles as well, whose magnetic charges must satisfy the DSZ condition  $q^{(1)} \cdot h^{(2)} - q^{(2)} \cdot h^{(1)} \in \mathbb{Z}$ . If we order the eigenvalues  $\phi_i > \phi_{i+1}$ , we expect the stable monopoles to be those with charges in successive factors:  $h = (0, \dots, 0, +1, -1, 0, \dots)$ , so we introduce the basis vectors  $h_m^i = \delta_m^i - \delta_{m+1}^i$ . The mass of a reduced multiplet is determined by its  $\mathcal{N} = 2$  central charge, which is determined by its electric and magnetic charges to be

$$M = \sqrt{2}|Z| = \sqrt{2}|a \cdot q + a_D \cdot h| \quad (2.4)$$

with  $a_{Di} = \partial\mathcal{F}/\partial a_i$ , a result motivated by duality as we discuss below.

The  $a_D$  associated with individual monopoles are  $a_{Dm} = h_m^i a_{Di}$ , whose inverse is  $a_{Di} = a_{D,i=1} - \sum_{m < i} a_{Dm}$ . We will always use the indices  $ij$  versus  $mn$  to distinguish the two bases. Finally, we introduce  $a_n = \sum_{i \leq n} a_i$  with inverse  $a_i = q_i^n a_n = a_{n=i} - a_{n=i-1}$ . These are canonically conjugate to the  $a_{Dm}$  in the sense that  $a_n = -\partial\mathcal{F}_D/\partial a_{Dn}$ .

We first discuss the naive semiclassical regime and its large  $N$  limit. To break  $SU(N)$  to  $U(1)^{N-1}$  and give every charged multiplet a mass of order  $N^0$  or greater, we need to choose  $\phi$  so that the minimum difference between eigenvalues is  $O(N^0)$ . A representative choice is  $\phi_{ij} = v(i - (N+1)/2)\delta_{ij}$ , where  $v$  is the characteristic scale of the vev. The classical prediction for the mass of the multiplet  $A_{ij}$  is then  $v|i - j|$  and we have a linearly rising spectrum with multiplicity  $O(N)$  at each mass level. Monopoles are expected to have masses  $4\pi Nv/g_0^2$  and do, although it may be interesting to note that the formula  $a_{Di} = \partial(\mathcal{F}_{cl} + \mathcal{F}_1)/\partial a_i$  produces unusual corrections to this, e.g. with  $\log N$  dependence.

We now turn to the exact solution [2,3,4].  $\mathcal{F}(A)$  is determined in the quantum theory by combining analyticity with a physical ansatz for the number and types of its singularities, at points where BPS saturated states become massless. What makes it

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\* We introduced the extra ‘ $e^3$ ’ compared with [3] to be consistent with an extra  $\frac{1}{2}$  in the curve (2.6) below. It will turn out that this simplifies the formulas at the massless monopole point.

possible to study the strong coupling regime and combine the information from different regimes is the existence of exact duality transformations on the effective Lagrangian, and the generalization of the Witten effect: encircling any singularity in moduli space produces a non-trivial  $Sp(2N-2; \mathbb{Z})$  transformation on the electric and magnetic charges of all states.

$\mathcal{F}(A)$  is not a single-valued function of  $A$  (as is already clear from (2.3)) and thus we must distinguish the coordinates on  $\mathcal{M}$  (for which we retain the name  $\phi_i$ ) from the scalar components  $(a_{Di}, a_i)$  of the fields in the effective Lagrangian. It turns out that  $\mathcal{F}(A)$  is most simply expressed in terms of an auxiliary Riemann surface  $\mathcal{C}$  which varies on  $\mathcal{M}$ , defined by the curve

$$y^2 = P(x)^2 - \Lambda^{2N}$$

$$P(x) \equiv \frac{1}{2} \det (x - \langle \phi \rangle) = \frac{1}{2} \prod_i (x - \phi_i). \quad (2.5)$$

(We generally set  $\Lambda = 1$  in the following). The  $(a_{Di}, a_j)$  are then integrals of the meromorphic form  $\lambda = (1/2\pi i)(x/y)dP(x)$  over a basis of one-cycles with the intersection form  $h_i \cdot q_j$ .

To define the one-cycles, order the branch points  $x_i$ . Let  $\gamma_i$  for  $1 \leq i \leq N$  encircle the branch points  $x_{2i-1}$  and  $x_{2i}$ , and  $\alpha_m$  for  $1 \leq m < N$  encircle the cut running from  $x_{2m}$  to  $x_{2m+1}$ . The  $\gamma_i$  are not independent but satisfy  $\sum_i \gamma_i = 0$ , while the  $\alpha_m$  are independent. Their intersection matrix is  $\langle \alpha_m, \gamma_j \rangle = \delta_{m,j} - \delta_{m,j+1}$  and thus we can associate the quarks with the cycles  $\gamma_i$  and the monopoles with  $\alpha_m$ . We also define a set of cycles conjugate to the  $\alpha_m$  as

$$\beta_n = \sum_{i \leq n} \gamma_i. \quad (2.6)$$

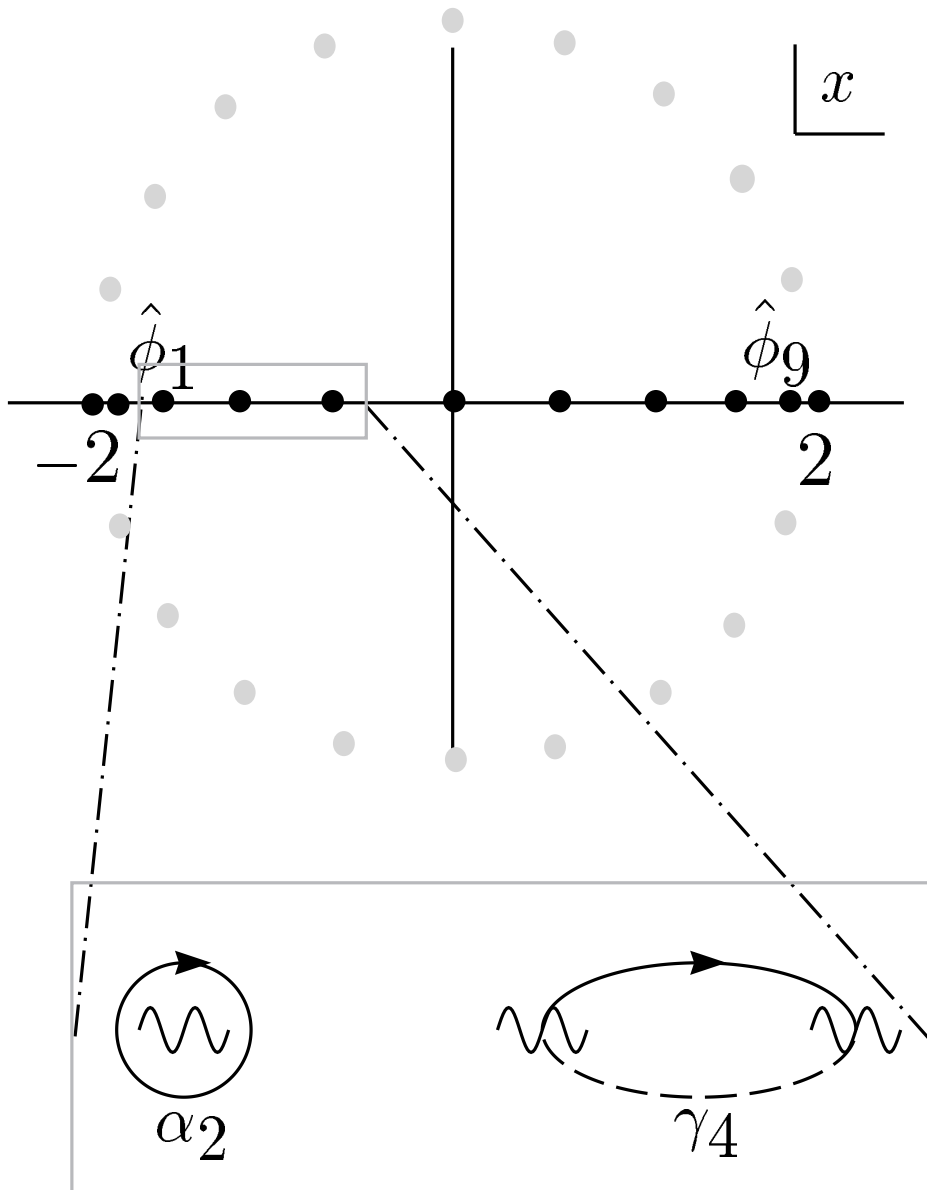
We then write

$$a_{Dm} = \oint_{\alpha_m} \lambda \quad a_n = \oint_{\beta_n} \lambda. \quad (2.7)$$

The strong coupling regime is controlled by points where monopoles and dyons become massless:  $\vec{h} \cdot \vec{a}_D + \vec{q} \cdot \vec{a} = 0$ . The vacua of the  $\mathcal{N} = 1$  theory will come from points in moduli space at which monopoles coupling to each  $U(1)$  become massless. There are  $N$  such points, with a simultaneous degeneration of all the  $\alpha$ -cycles of the quantum curve. In the realization  $y^2 = P(x)^2 - 1$  this will happen when the  $N$  cuts are lined up, each with one branch point coinciding with the next. In other words, we require  $P(x)^2 - 1$  to have  $N - 1$  double zeros and two single zeros. This condition can be satisfied using Chebyshev polynomials:

$$P(x) = \frac{1}{2} \det (x - \langle \phi \rangle) = T_N \left( \frac{x}{2} \right) = \cos \left( N \arccos \frac{x}{2} \right)$$

$$P(x)^2 - 1 = \left( \frac{x^2}{4} - 1 \right) U_{N-1} \left( \frac{x}{2} \right)^2. \quad (2.8)$$



We obtain  $N - 1$  more solutions by complex rotations  $x \rightarrow e^{i\pi r/N} x$ . Since each is associated with a ground state of the  $\mathcal{N} = 1$  theory, whose Witten index is  $N$ , we do not expect other solutions to exist.\*

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\* The full story is subtle as there are partial degenerations of the curve at which  $N - 1$  periods of  $\lambda$  vanish. These are limits in which more than two branch points coalesce, and an example

The eigenvalues of  $\phi$  are non-degenerate,  $\phi_n = 2 \cos \pi(n - \frac{1}{2})/N$ . They have spacing  $O(1/N)$  in the center of the “band” and  $O(1/N^2)$  at the edges. The double branch points of the curve are  $\hat{\phi}_n = 2 \cos \pi n/N$  for  $1 \leq n < N$  and the single branch points are at  $\pm 2$ .

Let us compute the periods of the maximally degenerate curve  $\mathcal{C}_0$ . Since the cuts are all on the real axis, the  $\alpha$  periods will be imaginary while the  $\gamma$  periods will be real. Changing variables from  $x = 2 \cos \theta$  to  $\theta$ , we have

$$\begin{aligned}
P(x) &= \cos N\theta \\
y &= i \sin N\theta \\
\lambda &= \frac{1}{2\pi i} \frac{x}{y} \frac{\partial P(x)}{\partial x} dx \\
&= \frac{N}{\pi} \cos \theta d\theta \\
\left. \frac{\partial \lambda}{\partial \phi_i} \right|_{\mathcal{C}_0} &= -\frac{1}{2\pi i} \frac{1}{y} \frac{\partial P(x)}{\partial \phi_i} dx + d(\dots) \\
&= \frac{1}{\pi} \frac{1}{x - \phi_i} \cot N\theta \sin \theta d\theta.
\end{aligned} \tag{2.9}$$

The  $a_{Dm}$  are integrals around the  $\alpha$  cycles. These degenerate to integrals around the  $\hat{\theta}_m$ . Since  $\lambda$  is non-singular, all  $a_{Dm} = 0$  for  $\mathcal{C}_0$ . When we vary the curve, we could get two types of contributions. First, the zeroes of  $P^2 - 1$  split. This will be important below but here we simply inflate the contour to enclose the new branch points. Second, the derivatives of  $\lambda$  have poles:

$$\begin{aligned}
B_{mi} \equiv -i \frac{\partial a_{Dm}}{\partial \phi_i} &= -\frac{i}{\pi} \oint_{\alpha_m} d\theta \frac{1}{x - \phi_i} \cot N\theta \sin \theta \\
&= \frac{2}{\phi_i - \hat{\phi}_m} \frac{\cos N\theta \sin \theta}{\frac{\partial}{\partial \theta} \sin N\theta} \Big|_{\theta=\hat{\theta}_m} \\
&= \frac{1}{N} \frac{\sin \hat{\theta}_m}{\cos \theta_i - \cos \hat{\theta}_m}
\end{aligned} \tag{2.10}$$

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is  $P(x) = \frac{1}{2}x^N - \Lambda^N$ . A coalescence of  $k$  branch points will cause the periods of  $\lambda$  on cycles surrounding any two branch points to vanish simultaneously. One can choose a basis of  $k - 1$  such cycles, and they have a non-zero intersection form, meaning that the associated massless particles would not be mutually local. One can show that at such points at least one  $U(1)$  will not couple to any of the massless particles and so will not confine when an  $\mathcal{N} = 2$ -breaking mass perturbation forces the massless particles to condense. Thus these are not candidates for  $\mathcal{N} = 1$  vacua, where full confinement is expected. We thank P. Argyres, W. Lerche and L. Randall for discussions on this point.



The matrix  $B_{mi}$  is simple in the following basis (appendix A):

$$\sum_{m=1}^{N-1} B_{mi} \sin \frac{\pi km}{N} = \cos \frac{\pi k(i - \frac{1}{2})}{N}. \quad (A.1)$$

The  $a_m$  are integrals around the  $\beta$  cycles. We find

$$\begin{aligned} a_m &= 2 \int_0^{\hat{\theta}_m} \lambda \\ &= \frac{2N}{\pi} \int_0^{\pi m/N} d(\sin \theta) \\ &= \frac{2N}{\pi} \sin \frac{\pi m}{N}. \end{aligned} \quad (2.11)$$

These determine the masses of BPS saturated electrically charged states. Quark masses would be given by the  $\gamma$  periods, which are differences of  $\beta$  periods:  $m_j = \frac{2\sqrt{2}N}{\pi} (\sin \frac{\pi j}{N} - \sin \frac{\pi(j-1)}{N})$ . For odd  $N$ ,  $m_{(N+1)/2}$  vanishes, but since there are no quarks in the theory at hand, this is not a difficulty. The ‘W bosons’ which are present have masses  $m_{ij} = m_i - m_j$ , which are all non-zero at finite  $N$ .

The first comment to make is that these masses are different for particles with charges in different  $U(1)$  factors. As we commented in the introduction, this is generically true at weak coupling: an explicit choice of Higgs vev will break the discrete gauge symmetry relating the factors. What may be surprising is that this persists for small Higgs vev and at the vacuum relevant for the related  $\mathcal{N} = 1$  gauge theory. There is simply no symmetry under permuting the periods of the curve  $\mathcal{C}_0$ .

Our second comment is that the mass of the lightest  $W$  boson goes to zero in the large  $N$  limit as  $m_{12} = \Lambda\pi^2/N^2$ . This is important because it determines the energy scale at which our effective Lagrangian breaks down, and we discuss this point below.

The derivatives of the  $a_m$  are

$$\begin{aligned} A_{mi} &= \frac{\partial a_m}{\partial \phi_i} = \frac{1}{2\pi} \oint_{\beta_m} d\theta \frac{\sin \theta}{\cos \theta - \cos \theta_i} \cot N\theta \\ &= \frac{1}{\pi} \int_0^{\pi m/N} d\theta \frac{\sin \theta}{\cos \theta - \cos \theta_i} \cot N\theta \end{aligned} \quad (2.12)$$

This is log divergent, as it should be on physical grounds. The dual gauge coupling in each  $U(1)$  factor has a positive (IR free) beta function produced by a one-loop diagram involving its light monopole. Away from the massless monopole point, it will be cut

off at the mass of the monopole to produce the same result in each factor as in [3]:<sup>\*</sup>  
 $4\pi/e_D^2 \sim \tau_{mn}^D \sim -(i/2\pi)\delta_{mn} \log a_{Dm}$ .

To get a finite result one must perturb the curve slightly to give the monopoles small masses. This again splits the double zeroes and modifies  $\lambda$ . For calculating the coefficient of the logarithm, the modification of  $\lambda$  does not matter, and the result is given by integrating  $(\partial\lambda/\partial\phi_i)_{\mathcal{C}_0}$  up to the branch point, which we parametrize as  $\theta \equiv \hat{\theta}_m - \epsilon_m/N$ . The integral then simply gives an endpoint divergence,

$$\begin{aligned} A_{mi} &\sim -\frac{1}{\pi N} \frac{\sin \theta}{\cos \theta_i - \cos \theta} \log \sin N\theta \Big|_{\theta=0}^{\theta=\pi m/N - \epsilon_m/N} \\ &\sim -\frac{1}{\pi} \log \epsilon_m B_{mi}. \end{aligned} \quad (2.13)$$

The  $\epsilon_m$  are computed by varying the equation  $y^2 = P(x)^2 - 1$ . Since we are splitting double zeroes we will find  $\epsilon \sim (\delta\phi)^{1/2}$ . Using (2.9) we have

$$\begin{aligned} 0 &= (\delta y)^2 + 2P \sum_i \frac{\partial P}{\partial \phi_i} \delta \phi_i \\ &= -\sin^2 N\delta\theta - 2\cos^2 N\theta \sum_i \frac{\partial P}{\partial \phi_i} \delta \phi_i \\ \epsilon_m^2 &= -2 \sum_i \frac{1}{\hat{\phi}_m - \phi_i} \delta \phi_i \\ &= \frac{N}{\sin \hat{\theta}_m} \sum_i B_{mi} \delta \phi_i \\ &= \frac{N}{\sin \hat{\theta}_m} \delta a_{Dm} \end{aligned} \quad (2.14)$$

using (2.10).

Thus the period matrix diverges as

$$\begin{aligned} \tau_{mn}^D &= \frac{\partial a_m}{\partial a_{Dn}} = -i \sum_i A_{mi} B_{ni}^{-1} \\ &\sim -\frac{i}{2\pi} \delta_{mn} \log \frac{a_{Dm}}{\Lambda_m} \end{aligned} \quad (2.15)$$

with  $\Lambda_m \equiv \Lambda \sin \hat{\theta}_m/N$ .<sup>\*</sup> This checks with the expectations for the beta function, and confirms the existence of one monopole in each factor (there is no overall  $N$ ).

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<sup>\*</sup> The  $i/\pi$  of [2] is in different charge conventions.

<sup>\*</sup> The preceding calculation of  $A_{mi}$  left out constant terms coming from the bulk of the integral, which could change  $\Lambda_m$ . In section 5 we will do this more carefully and show that important constants are present, but are compatible with the definition of  $\Lambda_m$  we give here.

At this order the different  $U(1)$  factors are completely decoupled. Note that the decoupled factors are different at weak and at strong coupling ( $\tau$  is diagonal in a different basis). Physically, the two limits are controlled by different light degrees of freedom. The  $m$ 'th monopole beta function turns on at the scale  $\Lambda_m$ , and at sufficiently low energy, their contribution  $\delta_{mn}/e_D^2$  will dominate any structure in  $\tau$  from higher energies, and decouple the factors.

In general, an extended object of size  $L$  does not contribute to the beta function at energies  $E > 1/L$ , so we can interpret the scale  $\Lambda_m$  as an indication of the size of the monopole. Semiclassically the monopole size would have been  $L \sim 1/m_W$  and even though here gauge couplings are  $O(1)$ , we still have  $\Lambda_m = m_{m,m+1}/2\sqrt{2}\pi$  with  $m_{m,m+1}$  to good accuracy the lightest  $W$  mass coupling to that factor.

The effective Lagrangian around the maximal degeneration is

$$\begin{aligned} \mathcal{L}_{eff} = \sum_m \bigg[ & \text{Im} \frac{i}{e_{Dm}^2} \left( \int d^4\theta A_D^m \bar{A}_D^m + \int d^2\theta (W_D^m)^2 \right) \\ & + \text{Im} \int d^4\theta M_m^+ e^{V_{Dm}} M_m + \tilde{M}_m^+ e^{-V_{Dm}} \tilde{M}_m \\ & + \text{Re} \sqrt{2} \int d^2\theta A_{Dm} M_m \tilde{M}_m \bigg]. \end{aligned} \quad (2.16)$$

The  $U(1)$  factors will couple at higher orders in an expansion of the kinetic term, but we expect these interactions to be suppressed by powers of  $p/\Lambda_m$ . One physical source of this coupling is loops of massive  $W$  particles charged under more than one  $U(1)$ , and such interactions will be suppressed by the mass of the  $W$ s.

An interesting question for contact with the standard large  $N$  limit (to be discussed below) is whether the interactions are suppressed in the standard way, as  $\phi^{2+k}/N^k$ . It is obviously not so for the superpotential terms in (2.16), which are completely determined by  $\mathcal{N} = 2$ . Nor is there an obvious reason for it to be true of the couplings between  $U(1)$  factors.

A  $\mathcal{N} = 1$  theory can be obtained by adding the perturbation  $W = Nm \text{Tr} A^2$  to the bare superpotential. Following [2] we interpret this in the effective Lagrangian as the perturbation  $W = Nm \sum_i \phi_i^2$ , the observable (single-valued on moduli space) equivalent to  $Nm \text{Tr} A^2$  in the semiclassical limit, and thus the exact superpotential is

$$W = \sqrt{2} \sum_m A_{Dm} M_m \tilde{M}_m + Nm \sum_i \phi_i^2. \quad (2.17)$$

To calculate its effect we need to change variables from  $\phi_i$  to  $a_{Dm}$ . We find that the vacua  $W' = 0$  satisfy

$$\begin{aligned}\sqrt{2}\langle M\tilde{M}\rangle_n &= 2Nm \sum_i (B^{-1})_{ni}\phi_i \\ \sum_n B_{ni}\langle M\tilde{M}\rangle_n &= \sqrt{2}Nm\phi_i\end{aligned}\tag{2.18}$$

and  $a_{Dn} = 0$ . This means of course that the kinetic term cannot literally be given by (2.15). Physically, the monopole loop integrals are now cut off by masses produced by the  $\mathcal{N} = 1$  part of the superpotential. We use the prescription of taking  $a_{Dn} = \langle M\tilde{M}\rangle_n^{1/2}$  in the  $\mathcal{N} = 2$  kinetic term to account for this.

The equation (2.18) has the solution

$$\langle M\tilde{M}\rangle_n = 2\sqrt{2}Nm \sin \frac{\pi n}{N}\tag{2.19}$$

and all of the  $U(1)$  gauge symmetries are spontaneously broken. We did not find an explicit discussion of the particular  $\mathcal{N} = 2$  abelian Higgs theory which appears here in the literature, but it is easy to work out\* that the particles in a given  $U(1)$  factor all have the same mass

$$m_n^2 = 2e_{Dn}^2 \langle M\tilde{M}\rangle_n\tag{2.20}$$

and the string tension in the factor is

$$\kappa_n = 2\pi \langle M\tilde{M}\rangle_n.\tag{2.21}$$

It is different in each factor, and roughly proportional to the lightest  $W$  mass in the factor,  $\kappa_n \propto mN^2 m_{n,n+1}$ . The gap is non-vanishing in the large  $N$  limit. Because the scalar and vector retain the same mass after  $\mathcal{N} = 2$  breaking, there is no long range potential between two strings (they are ‘neutrally stable’).

Using our prescription to determine the kinetic term,

$$\begin{aligned}\frac{4\pi}{e_{Dn}^2} &\sim -\frac{1}{4\pi} \log\left(\frac{m\Lambda}{\Lambda_n^2} N \sin \frac{\pi n}{N}\right) \\ &= -\frac{1}{4\pi} \log\left(\frac{mN^3}{\Lambda \sin \frac{\pi n}{N}}\right)\end{aligned}\tag{2.22}$$

and there is a weak (logarithmic) variation between the couplings in the different  $U(1)$  factors.

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\* One can simplify one’s life by observing that the linear part of the  $\mathcal{N} = 2$  breaking term in  $W$  is a pure  $F$  term, which using  $\mathcal{N} = 2$  can be rotated into a  $D$  term. This turns the vev into  $\langle M^2 \rangle = 2\sqrt{2}Nm \sin \frac{\pi n}{N}$  and  $\langle \tilde{M} \rangle = 0$  and the string solution becomes that of the usual  $\mathcal{N} = 1$  abelian Higgs model.

### 3. Physics at finite $N$ .

The effective Lagrangian (2.16) with the superpotential (2.17) provides an explicit realization of the abelian monopole condensation model of confinement in an  $SU(N)$  gauge theory. Such models of confinement were much discussed previously, as in [6], on a qualitative level. However, what in retrospect seems an obvious question was not discussed to our knowledge: namely, how can an analysis in which the theory looks like a broken  $U(1)^{N-1}$  gauge theory avoid having  $N-1$  distinct types of flux tubes and a corresponding multiplicity in the spectrum?

In fact we see from (2.21) that the  $U(1)$  factors have differing string tensions. (There is a  $\mathbb{Z}_2$  symmetry  $a_i \rightarrow -a_{N+1-i}$ ,  $\langle M\tilde{M} \rangle_n \rightarrow \langle M\tilde{M} \rangle_{N-n}$  which reduces the number of distinct string tensions to  $\lfloor \frac{N-1}{2} \rfloor$ .) Although the individual factors cannot be studied in isolation, since the coupling between them is due to the finite mass  $W$  bosons, one can see physical effects of the ‘heavier’ factors.

The simplest example is the expectation value of a Wilson loop. At sufficiently long distances it is energetically favorable for the heavier string tensions to be screened by  $W$  bosons. The parameter which controls this is the ratio of string tensions to  $W$  masses  $\kappa/m_W^2 \sim N^2 m/\Lambda$ , and for any  $m/\Lambda \ll 1$  the almost- $\mathcal{N} = 2$  analysis should be valid. Thus in the area law regime  $\kappa L^2 \gg 1$ , the Wilson loop will show a crossover from an intermediate distance  $L < m_W/\kappa \sim 1/N^2 m$  behavior, the sum of terms  $\exp -\kappa_n L^2$  with distinct string tensions, to a long distance behavior governed by the lowest string tension.

The distinct string tensions are already visible for  $SU(3)$ . Since the quark charges expressed in the monopole basis are  $(1\ 0)$ ,  $(-1\ 1)$  and  $(0\ -1)$ , the intermediate range fundamental Wilson loop will go as  $2 \exp -\kappa_1 L^2 + \exp -2\kappa_1 L^2$ .

The underlying  $U(1)^{N-1}$  symmetry of the effective theory has other physical consequences as well. It is clearly visible in the light spectrum, the ‘glueballs’ described by the fields of our effective Lagrangian. These are weakly coupled because  $e_D$  is small (for  $m \ll \Lambda$ ), and the different  $U(1)$  factors are derivatively coupled (again, controlled by the  $W$  masses).

Heavier confined states will vary in each factor as well. Let us consider a  $q\bar{q}$  meson state. (We assume for illustrative purposes that the theory with quarks is qualitatively similar, which will be true for heavy enough quarks. It is even true for light quarks in some cases, for example  $SU(2)$  with one flavor. Of course one could talk about states containing gluinos in the present theory.) For each single  $q\bar{q}$  state found in a conventional analysis (for example the strong coupling expansion), it appears that we would find  $N$  states, distinguished by the charge of the quark, and bound with different string tensions.

The  $SU(N)$  gauge theory broken to  $U(1)^{N-1}$  theory has an  $S_N$  discrete gauge symmetry, and one might have thought that this symmetry would somehow remove the multiplicity. However, we found that in the model under study, the dynamics picked a vacuum  $\langle\phi_i\rangle$  which spontaneously breaks the symmetry. This affects the expectation values  $a_i$  and the superpotential (2.17), and thus the string tensions and the physically observable ‘glueball’ masses (2.20) all break the symmetry. The masses of the  $N$  meson states contain contributions from both the string tensions and the quark masses  $m_q + a_i$ , and (except possibly at special values of  $m$ ,  $m_q$  and  $N$ ), there are no degeneracies between them. This leaves no doubt that they are distinct physical states.

The  $N$  states are not distinguished by any conserved quantum numbers and thus the heavier states are unstable to decay into the lightest. This is mediated by the derivative couplings between the  $U(1)$  factors and thus a decay amplitude will be  $\mathcal{A} \sim (\Delta E)^2/m_W^2$ . From the string tensions we can estimate  $\Delta E \sim \sqrt{m\Lambda}$  and thus the decay rate  $\mathcal{A} \sim m/\Lambda$  is controlled by the same arbitrarily small parameter which allowed us to see the distinct string tensions.

The structure of  $N$ -fold split multiplets is a rather unexpected difference between the physics of ‘almost- $\mathcal{N} = 2$ ’ supersymmetric gauge theory and non-supersymmetric gauge theory. Although definitive results for pure  $\mathcal{N} = 1$  gauge theory (or large  $m$  in the present theory) do not exist, we know of no evidence that this structure would persist, and believe it does not. Rather, we believe it is associated with the special features of almost- $\mathcal{N} = 2$  supersymmetry, in particular the light scalar.

One indication of this is that the natural gauge-invariant operators which create these states distinguish the  $U(1)$  factors by using the scalar vev. They are the chiral superfields

$$O_k = \tilde{Q}P_k(\phi)Q. \quad (3.1)$$

The polynomials  $P_k(\phi) = c_k \prod_{i \neq k} (\phi - \phi_i) = 2c_k T_N(\phi/2)/(\phi - \phi_k)$  satisfy  $P_k(\phi_i) = \delta_{ik}$  and pick out a single quark charge.

We might try to identify the split multiplets as bound states with the light scalar. This is clearly an appropriate description for scalar mass  $m \gg \Lambda$ . Thus we might hypothesize that as we increase  $m$ , a split multiplet evolves smoothly into a tower of bound states with mass splittings of  $O(m)$  and decay rates  $O(1)$ .

It still may appear that there is a distinction between the picture provided by spontaneously broken  $U(1)^{N-1} \times S_N$  gauge theory, and the more conventional confining description, which allows only  $U(1)^{N-1} \times S_N$  singlet states. However, there is no invariant order parameter distinguishing the two phases, the Higgs phase with  $S_N$  spontaneously broken, versus the confining phase. This is true even in the theory containing fundamental

matter, because the center of  $S_N$  is trivial. In other words, the eigenvalues of the adjoint scalar themselves transform under the  $N$  (or fundamental) of  $S_N$  and every operator in the Higgs phase of the theory will correspond to an  $S_N$  singlet operator constructed by using the scalar.

Which description is physically more appropriate depends on the strength of fluctuations in the  $S_N$  sector. In the language of spontaneously broken gauge theory, a configuration contains domains distinguished by different orderings of the  $\langle\phi_i\rangle$ , and separated by domain walls. The discrete gauge symmetry has the consequence that the domain walls can end on additional string solutions, with energy scale set by the symmetry breaking  $a_i - a_j$ . The language of spontaneously broken gauge theory is appropriate at energies low compared to this scale, while at higher energies strings and domain walls can be created freely, fluctuations in the  $S_N$  are unsuppressed, and the confining language is more appropriate.

The condition for weak coupling in the  $S_N$  sector is the same condition,  $E \ll m_W$ , that we used earlier and motivated on general grounds of validity of the effective Lagrangian.

Baryons will also exist in the theory and it is amusing to note that the  $U(1)^{N-1}$  charges of the quarks are such that the flux tubes would form a chain running from the  $i$ 'th quark to the  $i+1$ 'st quark in series. This can be contrasted to other pictures proposed in the past such as a flux tube with a 'Y' junction. Assuming the  $W$  bosons exist as stable particles, such a state also exists but is heavier for small  $m$ .

#### 4. The large $N$ limit.

One of the original motivations for this work was to examine the  $N \rightarrow \infty$  limit of the theory, test the assumptions made in previous work, and evaluate the hypothesis that large  $N$  gauge theory is simpler than finite  $N$ . One question we need to address is the strength of the coupling in the theory. Simple probes of this are connected expectation values of the gauge invariant field strength  $\text{tr } F^2$ . Standard large  $N$  counting predicts  $\langle \text{tr } F^2 \text{ tr } F^2 \rangle_c \leq O(1)$ ,  $\langle \text{tr } F^2 \text{ tr } F^2 \text{ tr } F^2 \rangle_c \leq O(1/N)$  and so on. If we calculate these expectation values at very long distance we can use the effective Lagrangian (2.2) to evaluate them. The field strength will be determined by its abelian parts.

Let us first examine the naive semiclassical regime where the electric couplings are weak and conventional perturbation theory is valid. To leading order the calculation is

governed by the eigenvalues of the kinetic term coefficient  $\tau_{ij}$ . Call these eigenvalues  $\tau_i$ . It is not hard to see that in this regime  $\tau_i \geq O(N)$ . The two point function is given by:

$$\langle \text{tr } F^2 \text{ tr } F^2 \rangle_c \sim \sum_i \frac{1}{\tau_i^2} \leq O(1/N). \quad (4.1)$$

This clearly satisfies large  $N$  counting. But this calculation also illustrates a natural mechanism for its breakdown; namely, the possibility of any  $\tau_i$  getting small. But this is just what happens near points where monopoles get massless, and so we expect large  $N$  counting to fail there. Near these points the dual coupling is weak, so perturbation theory for these quantities and hence (4.1) is again reliable. This appears to be an example, of which several are already known, of the lack of commutativity of the large  $N$  and infrared limits of a theory. It is the hope in ordinary confining gauge theories that these limits will in fact commute.

We now can ask what is the domain of validity of the naive semiclassical regime. For concreteness we imagine moving on a line in the moduli space that smoothly connects the extreme semiclassical regime with the massless monopole point  $\mathcal{C}_0$ . (Such paths are discussed in more detail below.) We can look for the breakdown of the semiclassical regime by examining when the one loop approximation becomes comparable to its first nontrivial correction, the one instanton term which is of order  $\Lambda^{2N}$ .

Let us estimate the one instanton correction to  $a_{Dm} = \int_{\alpha_m} \lambda$ . The contour  $\alpha_m$  surrounds the cut connecting the branch points  $x_{2m}$  and  $x_{2m+1}$ . Semiclassically, i.e., for small  $\Lambda$ , each pair of branch points surrounded by a  $\gamma$  cycle are located very close to a zero of  $P(x)$ ,  $x_{2i-1}, x_{2i} \sim \phi_i$ . To the required accuracy we can ignore the  $x$  dependence in every factor of  $P$  except  $(x - \phi_m)$  and  $(x - \phi_{m+1})$ , replacing  $x$  in these other factors by either  $\phi_m$  or  $\phi_{m+1}$ . The difference in results provides a measure of the accuracy of the estimate. Choosing  $\phi_i$ , we write  $P(x) \sim \frac{1}{2}Q(x - \phi_i)(x - \phi_{i+1})$  where  $Q = \prod_{j \neq i, i+1} (\phi_i - \phi_j)$ . Using  $\lambda = \frac{1}{2\pi i} \frac{x}{y} \frac{\partial P(x)}{\partial x} dx$  we can write

$$a_{Dm} \sim \int_{\alpha_m} \frac{x(x - \frac{\phi_m + \phi_{m+1}}{2})dx}{\sqrt{(\frac{1}{2}(x - \phi_m)(x - \phi_{m+1}))^2 - \Lambda^{2N}/Q^2}}. \quad (4.2)$$

This is closely related to the  $SU(2)$  expression for  $a_{Dm}$  with the matching condition  $\Lambda_{SU(2)}^4 = \Lambda^{2N}/Q^2$ . The  $SU(2)$  one instanton amplitude is  $a_{Dm}^{1I} \sim \Lambda^4/(\phi_{m+1} - \phi_m)^3$ . So  $a_{Dm}^{1I}$  is given by (assuming  $|\phi_{m+1} - \phi_m| \gg \Lambda$ )

$$a_{Dm}^{1I} \sim \frac{\Lambda^{2N}}{Q^2(\phi_{m+1} - \phi_m)^3}. \quad (4.3)$$



In the semiclassical region we have taken  $\phi_{ij} = v(i - (N + 1)/2)\delta_{ij}$ . Inserting this in (4.3) we find roughly

$$a_{Dm}^{1I} \sim \frac{\Lambda^{2N}}{(N!)^2 v^{2N}} \sim \left(\frac{\Lambda}{Nv}\right)^{2N}. \quad (4.4)$$

As  $N \rightarrow \infty$ ,  $a_{Dm}^{1I} \rightarrow 0$ . Note this is not merely the  $e^{-N}$  suppression of instantons. Such a suppression alone (reflected in the factor  $\Lambda^{2N}$ ) would allow this term to become important for  $v \sim \Lambda$ . The extra  $N!$  makes this term negligible until  $v \sim 1/N$ . (Of course this estimate is too crude to detect possible power of  $N$  prefactors of  $a_{Dm}^{1I}$ .) More generally, one instanton corrections are negligible until the splittings between neighboring  $\phi_i$  are  $\sim 1/N$ . Physically, this suppression arises because of the  $N$  different scales that are necessary to give all charged particles at least  $O(N^0)$  masses. Instanton contributions can be understood as associated with specific  $SU(2)$  subgroups, and all of the mass scales appear in the matching condition determining the coupling in a given  $SU(2)$  subgroup. Thus the one loop expression for  $a_{Dm}$  remains exact except on a vanishingly small region of moduli space as  $N \rightarrow \infty$ .

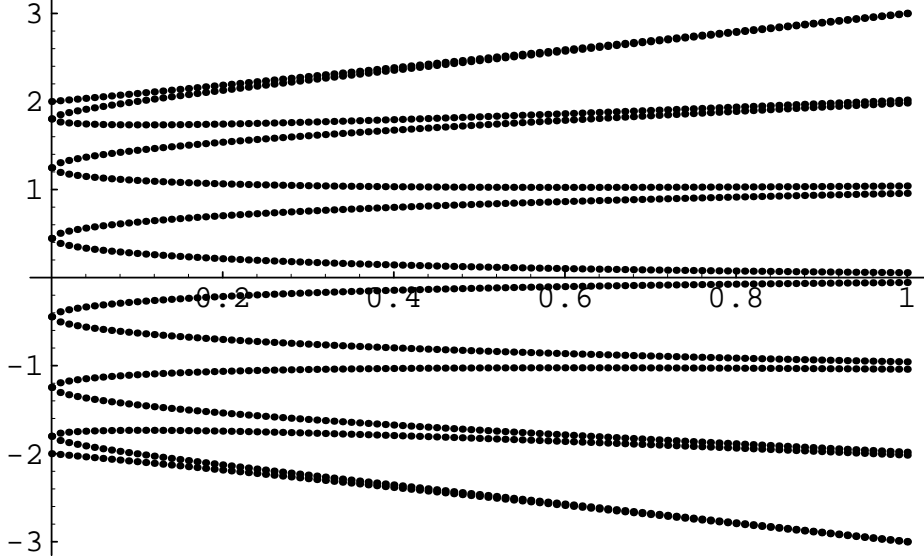
Of course this ‘vanishingly small’ region includes  $\mathcal{C}_0$  and hence is of vital interest for our study of confinement. So we now turn to the large  $N$  physics in the immediate neighborhood of  $\mathcal{C}_0$ . A striking feature we have already alluded to is the emergence of a very large hierarchy in scales near this point. The BPS mass formula gives the  $W_{ij}$  masses as  $m_{ij} = \sqrt{2}|a_i - a_j|$  with  $a_i = \frac{2N}{\pi}(\sin \frac{\pi i}{N} - \sin \frac{\pi(i-1)}{N}) \rightarrow 2 \cos \frac{\pi i}{N}$  at large  $N$ . The smallest mass is for  $W_{i,i+1}$  which for large  $N$  goes as  $m_{i,i+1} \rightarrow \frac{2\pi\sqrt{2}}{N} \sin \frac{\pi i}{N}$ . This is of order  $\sim 1/N$  for  $i$  near the center of the band, and  $\sim 1/N^2$  for  $i$  near the edge of the band.

The mass formula does not guarantee the existence of these  $W$  states. There are hyperplanes in moduli space on which particular BPS saturated states become unstable to decay, and thus the states need not be present in the spectrum on both sides [7,2]. When the ratio of the central charges  $Z(q, h) = a_i q^i + a_{Di} h^i$  associated with two particles becomes real,  $Z(q', h')/Z(q, h) \in \mathbb{R}$ , the particle with charges  $(q + q', h + h')$  is neutrally stable to decay into particles  $(q, h)$  and  $(q', h')$ . We should check that these light  $W$ s are present in the spectrum near  $\mathcal{C}_0$ . In fact they cannot be present at every point near  $\mathcal{C}_0$ . The light  $W$ ’s are nonlocal with respect to the massless monopoles  $M_m$  and their mass formulae are multivalued around the massless monopole point, as in [2].\* Thus for each light  $W$  there must be a hyperplane passing through  $\mathcal{C}_0$  on which it goes unstable.

This does not exclude the possibility that there exists a path connecting  $\mathcal{C}_0$  with the semiclassical regime on which the  $W$ ’s remain stable. We have found such a path,

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\* We thank N. Seiberg for pointing this out to us.



generalizing a similar path in [2] for  $SU(2)$ . It simply interpolates between the two regimes as follows:

$$\phi_i(t) = (1-t)\phi_i|_{\mathcal{C}_0} + t\left(\frac{N+1}{2} - i\right). \quad (4.5)$$

For  $N > 5$  and  $t \ll 1$  the double zeroes of  $P^2 - 1$  are split slightly and remain on the real axis. If we can show that for every  $t < 1$  the zeroes of  $P^2 - 1$  remain distinct, they must remain on the real axis, and the assignment of branch cuts and periods can be held fixed. We have shown this numerically for  $N \leq 21$ , and the behavior of the zeroes is sufficiently simple that we are convinced that this persists for arbitrary  $N$ . The accompanying plot shows the motion of the zeroes of  $P^2 - 1$  as a function of  $t$  for  $N = 7$ .

Thus the  $a_{Dm}$  will remain imaginary and the  $a_m$  will remain real for all  $t$ , it will be impossible for electrically charged states to decay into magnetically charged states for any  $t$ , and the presence of the  $W$ 's in the semiclassical regime implies the existence of light electrically charged states near  $\mathcal{C}_0$ . We have not yet proven the existence of a particular charged state. Because all  $a_m$  are real along the trajectory, all  $W$ 's are potentially neutrally stable. Nevertheless the lightest  $W$ 's will exist all along the trajectory. This follows from the fact that the ordering  $a_m > a_n$  for all  $m > n$  is preserved along the trajectory, for which the argument is simply that the  $a_m$  are real, so that if the ordering changes there must be a point in moduli space at which  $a_m = a_{m+1}$ , but this would imply the existence of a massless  $W$  and a singularity on moduli space not associated with a degeneration of the curve, which is impossible. Given this ordering, the particles  $W_{m,m+1}$  are necessarily stable all along the trajectory.

Although the  $W$  bosons clearly exist in the  $\mathcal{N} = 2$  theory, the question for the  $\mathcal{N} = 1$  theory is subtle, because we are working so close to a surface on which they go unstable. We cannot prove it with our present techniques, and will assume it is true.

Our effective Lagrangian is only valid for momenta small compared to the lightest  $W$  mass,  $p \leq 1/N^2$ . This is another indication of the noncommutativity of the large  $N$  and infrared limits in this theory. There should be a signature of this scale in the quantities we have already calculated, since the logarithmic running of the dual magnetic coupling at the edge of the band due to monopole loops should only start at scales below this mass. In fact we see in (2.15) precisely this effect.

There are a number of phenomena occurring near  $\mathcal{C}_0$ . First, as we have seen, instanton effects can become important roughly when eigenvalue splittings  $\Delta\phi \sim \Lambda/N$ . This indicates the initial breakdown of the semiclassical regime. Also, near  $\mathcal{C}_0$  the monopoles become light, at some point becoming comparable in mass to the lightest  $W$  they couple to. From (2.10) and the inverse of (A.1) we see that to give the monopoles such masses  $a_{Dm} \sim (\Lambda/N) \sin \pi m/N$ , we need to vary  $\mathcal{C}_0$  as  $\delta\phi_i \sim (\Lambda/N) \cos \pi(i - \frac{1}{2})/N$ . This is a variation of  $\phi_i$  which is  $O(1/N)$  compared to its value at  $\mathcal{C}_0$ . We see that the masslessness of the monopoles is due to delicate cancellations, a mathematical explanation of the appearance of multiple scales. In this region one can study the combined dynamics of electrically and magnetically charged particles as both their masses are made arbitrarily light with their ratio fixed. The first bit of this physics is the varying logarithmic cutoff of (2.15). A more detailed discussion will follow in the next section.

We now turn our attention to the confining phase by applying the  $\mathcal{N} = 2$  breaking perturbation  $Nm\text{Tr}A^2$  discussed earlier. The light  $W$ s dramatically affect the domain of  $m$  we can control. They cause flux tube breaking and coupling of the different  $U(1)$  sectors with no suppression at energy scales above their mass, making the theory unmanageable. So we are restricted to regime where the confining scale (e.g., a glueball mass) is lighter than the  $W$  scale. This requires taking  $m \sim \Lambda/N^4$ . Naively one might have expected to be able to take  $m \sim \Lambda$ . We see that as  $N \rightarrow \infty$  the abelian flux tube picture of confinement is valid on a vanishingly small part of the  $m$  axis.

In this region there is no reason to think that three Wilson loop expectation values are other than order one, and the glueball couplings described by the effective Lagrangian are manifestly so. If conventional large  $N$  physics holds in the pure  $\mathcal{N} = 1$  SYM theory obtained as  $m \rightarrow \infty$  a complicated but smooth rearrangement of the theory must take place. There certainly will be complicated dynamics occurring for  $m \geq \Lambda/N^4$ . Whether conventional expectations are realized we cannot say from this analysis. It is conceivable that order one couplings could persist as  $m \rightarrow \infty$ .

If conventional expectations are valid, they would probably hold until string tensions become of order light  $W$  masses. Since these masses go to zero as  $N \rightarrow \infty$  the region of validity of conventional expectations would occupy the whole  $m$  axis at  $N = \infty$ . There would be a sharp transition between the coulomb  $\mathcal{N} = 2$  theory described by one loop physics and the confining theory described by large  $N$  lore. The light abelian monopole region that smoothly interpolates between them would have disappeared.

## 5. A scaling trajectory.

So far, our discussion of the large  $N$  limit has been largely qualitative. We saw that as we move in the  $\mathcal{N} = 2$  moduli space towards  $\mathcal{C}_0$ , many things happen. We start with a simple  $U(1)^{N-1}$  gauge theory in which many calculations are exact at one loop. Gradually instantons turn on, and at the same time monopoles become light, eventually crossing the  $W$  masses to become the lightest degrees of freedom. They will then drive the beta function and cause the  $U(1)$  gauge factors to decouple in the monopole basis, different from the original basis in which  $\langle \phi \rangle$  was diagonal. Our analysis suggested that all of this happened in the scaling regime  $|\phi_i - \phi_{i+1}| \sim 1/N$ .

We will now study the relationship between these phenomena more systematically by introducing the following scaling trajectory for the eigenvalues of  $\langle \phi \rangle$ :

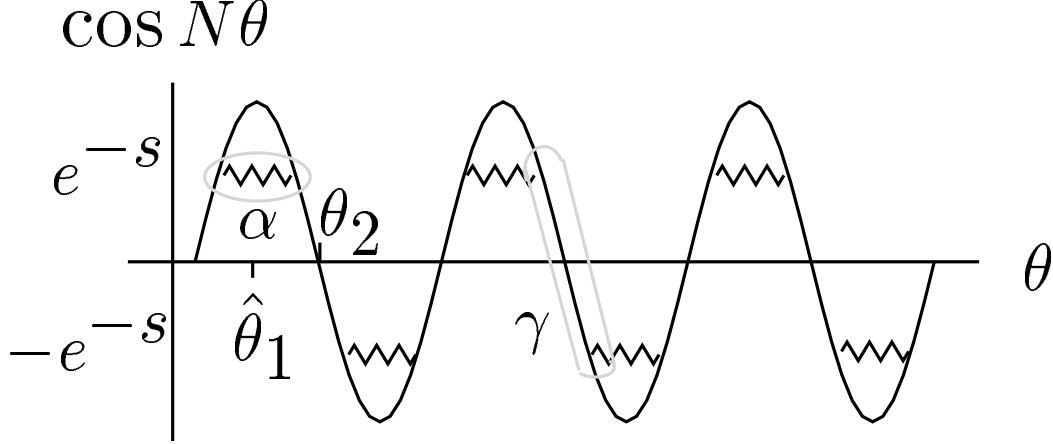
$$\phi_i(s) = e^{s/N} \phi_i|_{\mathcal{C}_0} = e^{s/N} 2 \cos \frac{\pi(i - \frac{1}{2})}{N} . \quad (5.1)$$

Introducing the scaled coordinate  $z = e^{-s/N} x$  we find the polynomial is of the form  $P(x, s) = e^s P(z)$  and the curve is given by  $y^2 = e^{2s}(P^2(z) - e^{-2s})$  where  $P(z) \equiv P(z)|_{\mathcal{C}_0}$ . As for our previous trajectory, the branch point evolution is smooth: taking  $z = 2 \cos \theta$  we have  $P(z) = \cos N\theta$  and the branch points are at  $\cos N\theta = \pm e^{-s}$ .

The trajectory is entirely in the scaling regime and even with  $s$  large the outlying  $|\phi_i - \phi_{i+1}|$  remain  $O(1/N^2)$ . However its essential property is that for  $s$  large, the length of the  $\gamma$  cycles shrinks to zero (in terms of  $x$ , as  $2e^{-s}$ ) and thus we recover semiclassical results.

The form  $\lambda$  becomes

$$\begin{aligned} \lambda &= \frac{e^{s/N}}{2\pi i} \frac{z dz}{\sqrt{P(z)^2 - e^{-2s}}} \frac{\partial P}{\partial z} \\ &= \frac{1}{\pi i} \frac{e^{s/N} \cos \theta d(\cos N\theta)}{\sqrt{\cos^2 N\theta - e^{-2s}}} . \end{aligned} \quad (5.2)$$



The large  $N$  simplification occurs because the integrals over the  $\alpha$  and  $\gamma$  cycles only cover a range of  $\theta \sim 1/N$ . Thus we can expand the  $\cos \theta$  factor to isolate the leading  $N$  dependence.

For  $a_i$  and  $a_{Dm}$  we obtain\* (using  $u = \cos N\theta$ )

$$\begin{aligned} a_i &= \frac{e^{s/N}}{\pi} \frac{2 \cos \theta_i}{\pi} \int_{-e^{-s}}^{e^{-s}} \frac{du}{\sqrt{e^{-2s} - u^2}} + O\left(\frac{1}{N^2}\right) . \\ &= e^{s/N} 2 \cos \theta_i \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} a_{Dm} &= \frac{4i \sin \hat{\theta}_m}{\pi} e^{s/N} \left( \frac{1}{N} I_1(s) + O\left(\frac{1}{N^3}\right) \right) \\ &= \frac{4i s e^{s/N}}{N} \sin \hat{\theta}_m . \\ I_1(s) &= \int_{u=e^{-s}}^{u=1} \frac{du \arccos u}{\sqrt{u^2 - e^{-2s}}} \\ &= s . \end{aligned} \quad (5.4)$$

We see that we can make monopoles much heavier than  $W$ 's along this trajectory. Monopole masses are comparable to the lightest  $W$  they couple to for  $s \sim 1$ . One loop contributions going like  $\log(a/\Lambda)$  will show  $s$  dependence  $\sim s/N$ , and we see from (5.4) that at leading order in  $N$  the monopole and  $W$  masses are given *exactly* by the one loop result, all the way to the massless monopole point! (We will see below that this is a special property of this trajectory and more generally trajectories  $\delta\phi_i \sim \cos \pi k(i - \frac{1}{2})/N$  with  $k \sim O(N^0)$ .)

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\* The integral  $I_1(s)$  is most easily evaluated by using partial integration to establish  $I_1(s) = -I_1(-s)$ . The only solution to this compatible with the analytic properties of  $I_1$  is  $I_1 = s$ . We are grateful to Alyosha and Sasha Zamolodchikov for providing this argument.

To compute  $\tau$ , we might use

$$\frac{\partial \lambda}{\partial \phi_i} = \frac{e^{s/N}}{2\pi} \frac{\sin \theta}{\cos \theta - \cos \theta_i} \frac{\cos N\theta}{\sqrt{\cos^2 N\theta - e^{-2s}}} d\theta. \quad (5.5)$$

There is a subtlety in the large  $N$  expansion of this form, due to the pole in the prefactor at  $\theta = \theta_i$ . Although the pole itself is cancelled by a zero of  $\cos N\theta$ , it is not justified to treat the prefactor as slowly varying.

We will deal with this by using the basis of variations

$$\frac{\partial \lambda}{\partial t_k} \equiv \sum_{i=1}^N \frac{\partial \lambda}{\partial \phi_i} \cos \frac{\pi k(i - \frac{1}{2})}{N}, \quad (5.6)$$

We will find that using this basis for ‘electric’ quantities and the basis  $\sin k\hat{\theta}_m$  for ‘magnetic’ quantities, the period matrices become simple and the kinetic term diagonal, for all  $s$  and at finite  $N$ .

Before doing this in detail, let us suggest why this gives simple results. From (A.1) and related formulas, it is clearly an appropriate basis for the massless monopole point  $s = 0$ . On the other hand, the large  $s$  limit is expected to reproduce the semiclassical results  $\partial a_i / \partial \phi_j = \delta_{ij}$  and  $\partial a_{Di} / \partial \phi_j = \partial^2 \mathcal{F} / \partial a_i \partial a_j$ . From (2.3) and (5.3) this is

$$\begin{aligned} \left. \frac{\partial a_{Di}}{\partial \phi_j} \right|_{\text{one loop}} &= \tau_{ij}^{(1)} = \frac{i}{2\pi} \left[ \delta_{ij} \sum_{k \neq i} t_{ik} - (1 - \delta_{ij}) t_{ij} \right] \\ t_{ij} &= \frac{2s}{N} + \log(2 \cos \theta_i - 2 \cos \theta_j)^2. \end{aligned} \quad (5.7)$$

The point is that the one-loop kinetic term is also diagonal in the cosine basis, by results similar to those in the appendix, making it possible for  $\tau$  along the entire trajectory to be diagonal. Let us estimate its eigenvalues in the large  $N$  limit: the difference  $t_{i+1,j} - t_{i,j} \sim B_{m=i,j} + O(1/N)$  and so (A.1) implies that  $t_{ij}$  will have eigenvalues  $t_\kappa \sim \frac{1}{2 \sin \pi k/N}$ . (There are corrections to this estimate, of  $O(k/N)$ ).

We proceed to compute the periods of

$$\begin{aligned} \frac{\partial \lambda}{\partial t_k} &\equiv \sum_{i=1}^N \frac{\partial \lambda}{\partial \phi_i} \cos \frac{\pi k(i - \frac{1}{2})}{N} \\ &= \frac{1}{2\pi} \frac{N \sin(N - k)\theta}{\sqrt{\cos^2 N\theta - e^{-2s}}} d\theta. \end{aligned} \quad (5.8)$$

We have ( $a = \arccos e^{-s}$ ,  $b = \arcsin e^{-s}$ ,  $\kappa = k/N$ )

$$\begin{aligned}
A_{jk}(s) &= \frac{\partial a_j}{\partial t_k} = \frac{(-1)^{j-1}}{\pi} \int_{-b}^b d\theta \frac{\sin((N-k)(\theta_j + \theta/N))}{\sqrt{e^{-2s} - \sin^2 \theta}} \\
&= \frac{1}{\pi} \int_{-b}^b d\theta \frac{\cos k\theta_j \cos(1-\kappa)\theta + \sin k\theta_j \sin(1-\kappa)\theta}{\sqrt{e^{-2s} - \sin^2 \theta}} \\
&= F(\kappa, s) \cos k\theta_j \\
F(\kappa, s) &= \frac{1}{\pi} \int_{-b}^b d\theta \frac{\cos(1-\kappa)\theta}{\sqrt{e^{-2s} - \sin^2 \theta}} \\
&\rightarrow \frac{1}{\pi} \sin \frac{\pi\kappa}{2} (-\log s) + 1 - O(\kappa) & s \rightarrow 0 \\
&\rightarrow 1 + \frac{\kappa(2-\kappa)}{4} e^{-2s} + \dots & s \rightarrow \infty
\end{aligned} \tag{5.9}$$

and similarly

$$\begin{aligned}
B_{mk}(s) &= \frac{\partial a_{Dm}}{\partial t_k} = G(\kappa, s) \sin k\hat{\theta}_m \\
G(\kappa, s) &= \frac{1}{\pi} \int_{-a}^a d\theta' \frac{\cos(1-\kappa)\theta'}{\sqrt{\cos^2 \theta' - e^{-2s}}} \\
&\rightarrow 1 + \frac{\kappa(2-\kappa)}{2} s + \dots & s \rightarrow 0 \\
&\rightarrow 2s \sin \frac{\pi\kappa}{2} + 1 - O(\kappa) + \dots & s \rightarrow \infty.
\end{aligned} \tag{5.10}$$

We can now compute  $\tau_{Dmn}(s)$ . Using (A.5) we see that  $\tau_{Dmn}(s)$  is diagonal in the  $\sin k\hat{\theta}_m$  basis:

$$\begin{aligned}
\tau_{Dmn}(s) &= \sum_k A_{mk} B_{kn}^{-1} = \sum_{i,k} q_{mi}^{-1} A_{ik} B_{kn}^{-1}, \\
\sum_n \tau_{Dmn}(s) \sin k\hat{\theta}_n &= \tau_{D\kappa}(s) \sin k\hat{\theta}_m
\end{aligned} \tag{5.11}$$

with eigenvalues

$$\tau_{D\kappa}(s) = \frac{1}{2 \sin \frac{\pi\kappa}{2}} \frac{F(\kappa, s)}{G(\kappa, s)}. \tag{5.12}$$

We verify that  $\tau_{D\kappa}(s) \rightarrow -\frac{1}{2\pi} \log s$ , independent of  $\kappa$ , as  $s \rightarrow 0$ . Similarly

$$\begin{aligned}
\tau_{ij}(s) &= \sum_{m,k} h_{im}^{-1} B_{mk} A_{kj}^{-1} \\
\sum_j \tau_{ij}(s) \cos k\theta_j &= \tau_{\kappa}(s) \cos k\theta_i \\
\tau_{\kappa}(s) &= \frac{1}{2 \sin \frac{\pi\kappa}{2}} \frac{G(\kappa, s)}{F(\kappa, s)}.
\end{aligned} \tag{5.13}$$

These formulae give an exact description of the physics along the scaling trajectory from the semiclassical domain all the way to  $\mathcal{C}_0$ , even at finite  $N$ .

Note the crucial fact that the eigenvalues  $\tau_\kappa(s)$  are *not* simply the inverses of the eigenvalues  $\tau_{D\kappa}(s)$ ! The change of basis from electric ( $ij$ ) to magnetic ( $mn$ ) couplings introduces an extra  $\frac{1}{4 \sin^2 \frac{\pi\kappa}{2}}$ . This will have important consequences and seems quite mysterious; it may be relevant that (for example) in the magnetic basis, the  $W$  bosons responsible for the one loop contribution to  $\tau$  have charges with alternating signs (such as  $(1 \ -1 \ -1 \ 1)$ ), and in summing their loop effects there will be large cancellations.

Let us consider the leading correction to the logarithm from the monopole beta function at small  $s$ , which come from the constant terms in  $F$  and  $G$ :

$$\tau_{D\kappa}(s) = -\frac{1}{2\pi} \log s + \frac{1}{2 \sin \frac{\pi\kappa}{2}} + O(1) + \dots \quad (5.14)$$

For  $k \sim O(N^0)$  ( $\kappa \sim 1/N$ ) and  $s \sim O(N^0)$ , this is a large correction, of  $O(N)$ . It is a threshold effect and eventually the monopole-induced running of the coupling will overwhelm it, but we need to go to scales  $s \sim e^{-N}$  for this to happen. Thus these effects are quite important at large  $N$ .

The same constants are responsible for the  $s$ -independent term at large  $s$  in

$$\tau_\kappa(s) = 2s + \frac{1}{2 \sin \frac{\pi\kappa}{2}} + O(1) + \dots \quad (5.15)$$

We see by comparison with (5.7) that this is just  $\tau_{\text{one loop}}$ . Rather magically, the low  $k$  ‘modes’ manage to be simultaneously weak in electric and magnetic variables.

Since the constant term in  $\tau_{D\kappa}$  has the same form, the form of  $\tau_D$  in the monopole basis can be inferred by analogy to  $\tau$ :

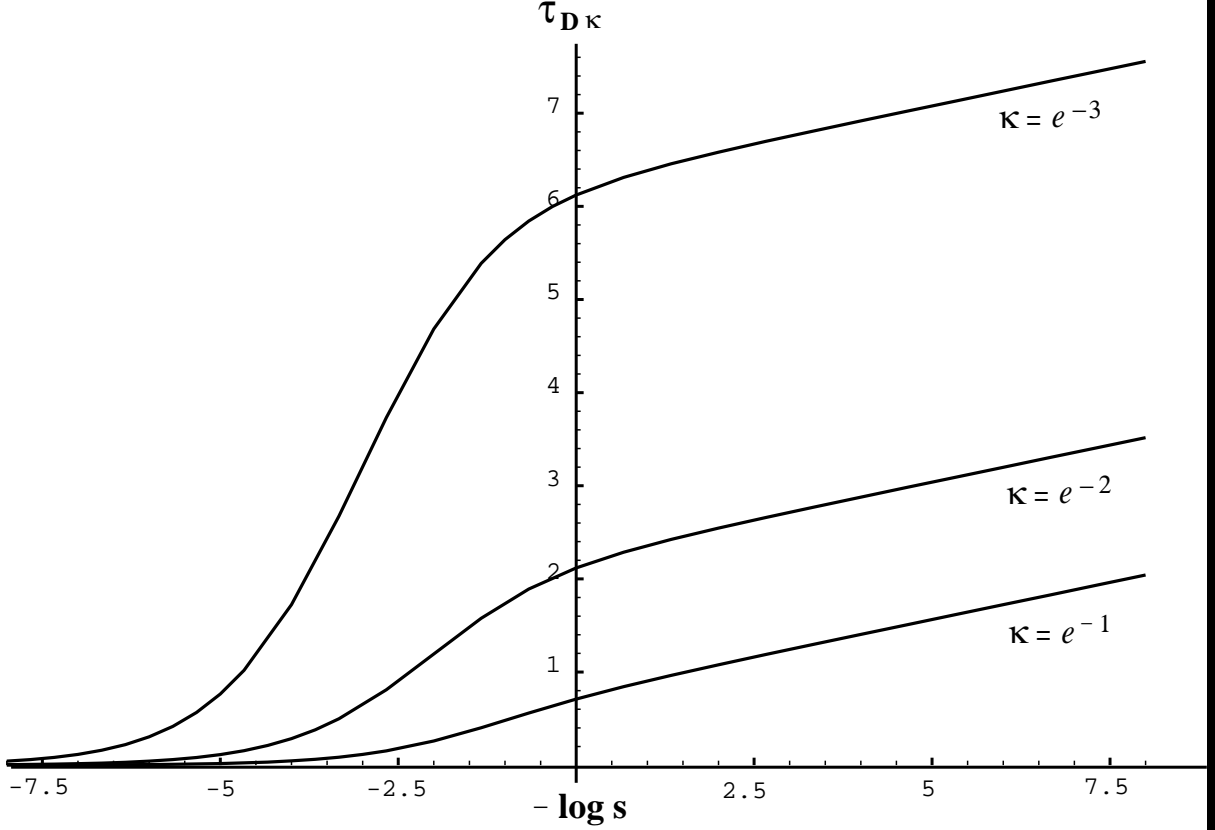
$$\tau_{Dmn} \sim -\frac{i}{2\pi} \delta_{mn} \log s + \frac{i}{2\pi} \log(2 \sin \hat{\theta}_m - 2 \sin \hat{\theta}_n)^2 + O(1) + \dots \quad (5.16)$$

This matrix is not diagonal and so a ‘magnetic observer’ would see peculiar transitions between the  $m$  quanta. This observer might posit the existence of light solitons carrying dual charge in four  $m$  factors (as mentioned above) that would mediate these transitions.

The similarity of (5.16) to the one-loop electric  $\tau$  is very intriguing and leads us to wonder if there is any sense in which it is a one-loop effect. The theory is certainly much more ‘self-dual’ than one would have guessed.

We now plot the scaling functions to illustrate their structure. Note that the  $-\frac{1}{2\pi} \log s$  behavior characteristic of the monopole one loop beta function sets in around  $s = 1$  uniformly in  $\kappa$ . This is just at the energy scale set by the inverse ‘size’ of the monopole, as





discussed earlier. The large residual  $s$  independent term in (5.14) is responsible for the  $\kappa$  dependent offsets of the curves. It is clear from the graph that one must go to  $s \sim e^{-1/\kappa}$  before this offset is negligible.

The off diagonal terms in (5.16) influence the energies of states in the confining phase obtained by adding the  $\mathcal{N} = 2$  breaking perturbation  $Nm\text{Tr}A^2$ . The basic structure of the flux tubes is determined by the monopole condensate which is determined from the superpotential alone and hence does not depend on  $\tau_{Dmn}$ . It is still given by (2.19). The string tensions do depend on  $\tau_{Dmn}$  and will be affected by the mixings described by (5.16). We have not yet studied this problem, but we have examined the closely related problem of the dual photon masses, by numerically diagonalizing the mass matrix. We found that the quantitative relation (2.20) is altered but that the scales of the masses remain unchanged. We expect the same for the string tensions. Note that these changes are significant for  $m \sim e^{-N}$ , far below the energies where string breaking due to light  $W$  pairs occur ( $m \sim 1/N^4$ ).

We remind the reader that some quantities ( $a_i$  and  $a_{Dm}$ ) were exact at one loop along the scaling trajectory. This occurred because the corrections to the one-loop results for their derivatives, from (5.9)-(5.10), were suppressed by powers of  $\kappa$ , so would happen for

any trajectory with  $\kappa \sim 1/N$ . If the same thing happened in other large  $N$  (supersymmetric) theories, for even a single trajectory into the true ground state (here the massless monopole point), it would have the important consequence that the ground state could be found just knowing the one-loop effective action.

## 6. Conclusions

Using the solution of  $\mathcal{N} = 2$  supersymmetric  $SU(N)$  gauge theory of [2,3,4], we have given an overview of some of the physics visible in the low energy effective Lagrangian. In particular we studied the  $\mathcal{N} = 1$  theory obtained by giving a mass to the chiral superfield, and extended Seiberg and Witten's explicit derivation of a monopole condensation model of confinement for  $SU(2)$  to  $SU(N)$ .

The Lagrangian relevant for this confining theory is a weakly coupled  $U(1)^{N-1}$  gauge theory with one light monopole hypermultiplet in each factor. The  $S_N$  discrete gauge symmetry that permutes the  $U(1)$  factors is *spontaneously broken*. This phenomenon pervades the physics of this system. In particular, there is a spectrum of different string tensions in the different factors,  $\kappa_n \sim m\Lambda N \sin \frac{\pi n}{N}$  and a spectrum of different light  $W$  boson masses  $\sim \frac{\Lambda}{N} \sin \frac{\pi n}{N}$  that connect the factors. This differs in many ways from our expectations about the pure  $\mathcal{N} = 1$  theory. It is possible for a dramatic but smooth rearrangement to take place for  $m \sim \Lambda$  that will connect these two descriptions. Perhaps this rearrangement can be understood as a smooth crossover from a regime of spontaneously broken  $S_N$  gauge symmetry to one where the  $S_N$  gauge symmetry is 'confined.'

We studied the large  $N$  limit in detail. The hierarchy of scales present for finite  $N$  becomes very large, as the lightest  $W$  mass  $\sim \Lambda/N^2$ . We found one signature of these very light particles in the monopole 'size' which controls the energy scale of onset of perturbative monopole coupling constant renormalization. Another very surprising signature is the off-diagonal terms in (5.16) which cause a coupling of the different magnetic factors. Understanding this phenomenon from the magnetic point of view, presumably as the result of light electric solitons in a weakly coupled magnetic theory, would be very interesting.

At large  $N$  the one loop result becomes exact almost everywhere on the  $\mathcal{N} = 2$  moduli space. We identified a scaling regime very close ( $\sim 1/N$ ) to the massless monopole point  $\mathcal{C}_0$  where instanton and monopole effects survive the large  $N$  limit and were able to give simple exact formulas for monopole and  $W$  masses and coupling constants along a scaling trajectory through this regime.

The coupling almost everywhere on the  $\mathcal{N} = 2$  moduli space is  $1/N$ , as expected, but near  $\mathcal{C}_0$  becomes large, contradicting standard large  $N$  lore. The infrared divergence of the

effective  $U(1)$  couplings, which does not commute with the large  $N$  limit, is the source of this. Such a phenomenon would seem to apply to any continuous monopole condensation picture of confinement, supersymmetric or otherwise. The region of this violation goes to zero as  $N \rightarrow \infty$ .

In the confining ‘almost  $\mathcal{N} = 2$ ’ theory, the abelian flux tube picture breaks down because of  $W$  pair creation when confining scales become comparable to  $W$  scales. This occurs for  $m \sim \Lambda/N^4$ , i.e., a vanishingly small region of the  $m$  axis. Conventional large  $N$  lore could be recovered in a smooth, complicated crossover we cannot control. As  $N \rightarrow \infty$  the region described by nontrivial light monopole physics shrinks to zero, possibly leaving an abrupt transition between one loop physics and large  $N$  pure  $\mathcal{N} = 1$  SYM behavior.

Some features of this model are reminiscent of standard hadronic phenomenology while others look very different (such as the weakly coupled glueballs), but overall we find the results encouraging for the idea of using supersymmetric gauge theory as a solvable starting point for attempts to model QCD dynamics. Surely interesting supersymmetric analogs of many other problems of strongly coupled gauge theory and QCD physics can be found, and we believe it will be very fruitful to study them using these techniques.

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We are informed by Greg Moore that he and Mans Henningson have independently obtained some of these results.

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## Appendix A. Trigonometric identities.

In section 2 we use

$$\sum_{m=1}^{N-1} B_{mi} \sin \frac{\pi k m}{N} = \cos \frac{\pi k (i - \frac{1}{2})}{N} \quad (\text{A.1})$$

which can be checked by writing ( $z_m = e^{i\hat{\theta}_m}$ ,  $w_i = e^{i\theta_i}$ )

$$B_{mi} = \frac{1}{iN} \left( \frac{1}{1 - w_i z_m} - \frac{1}{1 - w_i z_m^{-1}} \right). \quad (\text{A.2})$$

We convert the sum to run from 0 to  $2N - 1$ , for which  $\sum_m e^{\pi i k m / N} = 2N \delta_{k,0(\text{mod } 2N)}$ , by taking a sum  $\sum_{m=1}^{N-1}$  involving the second term in (A.2) and taking  $m \rightarrow 2N - m$ . Then

$$\begin{aligned} \sum_{m=1}^{N-1} B_{mi} \sin \frac{\pi k m}{N} &= \frac{1}{2N} \sum_{m=1}^{2N-1} \frac{w_i^{-1}}{w_i^{-1} - z_m} \left( e^{-i\pi k m / N} - e^{i\pi k m / N} \right) \\ &= \frac{w_i^k}{1 - w_i^{2N}} - \frac{w_i^{2N-k}}{1 - w_i^{2N}} \\ &= \frac{w_i^k + w_i^{-k}}{2}. \end{aligned} \quad (\text{A.3})$$

Using (A.1), the orthogonality of the basis  $\sin \pi k m / N$  under  $\sum_{m=1}^{N-1}$ , and the orthogonality of the basis  $\cos \pi k(i - \frac{1}{2})/N$  under  $\sum_{i=1}^N$ , we also have

$$\sum_{i=1}^N B_{mi} \cos k\theta_i = \sin k\hat{\theta}_m. \quad (\text{A.4})$$

In section 5 we use the change of basis

$$\sum_m q_i^m \sin k\hat{\theta}_m = \sin k\hat{\theta}_{m=i} - \sin k\hat{\theta}_{m=i-1} = 2 \sin \frac{k\pi}{2N} \cos k\theta_i \quad (\text{A.5})$$

We next evaluate the sum

$$f_k(\theta) = \frac{1}{N} \sum_{j=1}^N \frac{\sin \theta}{\cos \theta_j - \cos \theta} \cos k\theta_j = \frac{\sin(N-k)\theta}{\cos N\theta}. \quad (\text{A.6})$$

First, for  $k = 0$ , the sum is

$$\frac{1}{N} \frac{\partial}{\partial \theta} \log \prod_j (\cos \theta_j - \cos \theta) = \frac{1}{N} \frac{\partial}{\partial \theta} \log P(x) = \tan N\theta. \quad (\text{A.7})$$

For  $k$  integer, the sum is a periodic function of  $\theta$  with residues  $\cos k\theta$  at the points  $\theta = \theta_j$ , so it must take the form

$$f_k(\theta) = \tan N\theta \cos k\theta + \text{finite}. \quad (\text{A.8})$$

From the above, the finite part is known at the points  $\theta = \hat{\theta}_m$  to be  $-\sin k\theta$ . This is true of the function

$$f_k(\theta) = \tan N\theta \cos k\theta - \sin k\theta \quad (\text{A.9})$$

(equal to (A.6)) plus any periodic function vanishing at all of the  $\hat{\theta}_m$ , such as  $\sin 2Nl\theta$ . Such terms can be excluded by considering the derivative  $d/d\theta$  at  $\theta = 0$ .

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